# Infinite generation of the kernels of the Magnus and Burau representations

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**Abstract**

Consider the kernel $\mathrm{Mag}_g$ of the Magnus representation of the Torelli group and the kernel $\mathrm{Bur}_n$ of the Burau representation of the braid group. We prove that for $g \geq 2$ and for $n \geq 6$ the groups $\mathrm{Mag}_g$ and $\mathrm{Bur}_n$ have infinite rank first homology. As a consequence we conclude that neither group has any finite generating set. The method of proof in each case consists of producing a kind of "Johnson-type" homomorphism to an infinite rank abelian group, and proving the image has infinite rank. For the case of $\mathrm{Bur}_n$, we do this with the assistance of a computer calculation.

## 1 Introduction

**The Magnus kernel.** Let $S := S_{g,1}$ be a compact, connected, oriented surface of genus $g \geq 2$ with one boundary component. Let $\mathrm{Mod}_{g,1}$ denote the *mapping class group* of $S$, which is the group of homotopy classes of orientation-preserving homeomorphisms of $S$ which fix $\partial S$ pointwise. Let $\mathcal{I}_{g,1}$ denote the *Torelli group*, which is the subgroup of $\mathrm{Mod}_{g,1}$ consisting of elements that act trivially on $H := H_1(S, \mathbb{Z})$.

$\mathrm{Mod}_{g,1}$ acts on the fundamental group $\pi_1(S)$, inducing an action on the solvable quotient $\Gamma/\Gamma^3$, where $\Gamma := \pi_1(S)$, $\Gamma^2 = [\Gamma, \Gamma]$ and $\Gamma^3 = [\Gamma^2, \Gamma^2]$ are the first three terms of the derived series of $\Gamma$. In this paper we consider the group

$$\mathrm{Mag}_g := \mathrm{kernel}(\mathrm{Mod}(S) \to \mathrm{Aut}(\Gamma/\Gamma^3)).$$

In 1939, Magnus ([Ma]; see also [Bi, Chapter 3]) used the Fox calculus to construct a representation

$$r\colon \mathcal{I}_{g,1} \to \mathrm{GL}_{2g}(\mathbb{Z}H)$$

now called the *Magnus representation*. It follows from [Fox, Theorem 4.9] that the kernel of $r$ coincides with $\mathrm{Mag}_g$. This group is called the *Magnus kernel*.

It was an open question for some time whether or not $\mathrm{Mag}_g$ is nontrivial. This was settled in the affirmative by Suzuki in [S1]. The first main result of this paper is that $\mathrm{Mag}_g$ is in fact quite large.

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**Theorem 1.1.** *For $g \geq 2$ the group $H_1(\mathrm{Mag}_g, \mathbb{Z})$ has infinite rank.*

As the abelianization of a finitely-generated group has finite rank, we deduce the following.

**Corollary 1.2.** For $g \geq 2$ the group $\mathrm{Mag}_g$ has no finite generating set.

The idea of our proof of Theorem 1.1 is to define a kind of "Johnson-type" homomorphism (see [J1]):

$$\Psi\colon \ \mathrm{Mag}_g \to \mathrm{Hom}\left(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\right)$$

where $G = [\Gamma, \Gamma]$ and $G^{\mathrm{ab}}$ denotes the abelianization of $G$. We then construct infinitely many linearly independent elements contained in the image.

**The Burau kernel.** Let $B_n$ denote the braid group on $n$ strands. $B_n$ can be realized (see Section 4 below) as a subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of the free group of rank $n$. The *Burau representation* is a homomorphism

$$\rho_n\colon \ B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}]).$$

We define the *Burau kernel*, denoted $\mathrm{Bur}_n$, to be the kernel of $\rho_n$. Let $K$ be the kernel of the homomorphism $F_n \to \mathbb{Z}$ taking each fixed generator of $F_n$ to 1. It follows easily from [Fox] that

$$\mathrm{Bur}_n = \mathrm{kernel}(B_n \to \mathrm{Aut}(F_n/[K, K])).$$

While $\rho_3$ is faithful, it was a longstanding problem as to whether or not $\rho_n$ is faithful (i.e. whether $\mathrm{Bur}_n$ is nontrivial) for $n > 3$. This was solved by Moody [Mo], Long–Paton [LP], and Bigelow [Big] in various cases, with the result that $\mathrm{Bur}_n$ is nontrivial for $n \geq 5$; the case of $n = 4$ is still open. Our next main result is that $\mathrm{Bur}_n$ is in fact quite large for $n \geq 6$.

**Theorem 1.3.** *For $n \geq 6$ the group $H_1(\mathrm{Bur}_n, \mathbb{Z})$ has infinite rank; in particular, $\mathrm{Bur}_n$ has no finite generating set.*

To prove Theorem 1.3 we construct, similarly to the proof of Theorem 1.1 above, a homomorphism

$$\Phi\colon \ \mathrm{Bur}_n \to \mathrm{Hom}\left(K^{\mathrm{ab}}, \textstyle\bigwedge^2 K^{\mathrm{ab}}\right).$$

The elements which have been constructed in the kernel of the Burau representation are geometrically elegant, but algebraically very complicated; for example, the element of $\mathrm{Bur}_7$ found by Long–Paton can be described by a single diagram, but as a free group automorphism sends generators of $F_7$ to words of length up to 475137. Thus we need the assistance of a computer in order to calculate $\Phi$ explicitly (see Section 4 below for a full discussion). For the computations in this paper we use a simpler element $\phi_B \in \mathrm{Bur}_n$ for $n \geq 6$ found by Bigelow, which takes generators to words of length no more than 9841. Once we compute the form of $\Phi(\phi_B)$, we then use an equivariance property of $\Phi$ to prove that the image of $\Phi$ has infinite rank, from which Theorem 1.3 follows.

We remark that, as Problem 6.24 of [Mor], Morita posed the problem of determining the kernel of the Magnus and Burau (among other) representations. Theorem 1.1 and Theorem 1.3 can be viewed as a partial answer to this problem.

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## 2 Defining the homomorphisms

The following construction works whenever one considers a group of automorphisms of the universal 2-step nilpotent quotient of a group $G$ acting trivially on its abelianization. Johnson [J1] considered the case $G = \Gamma = \pi_1(S)$.

With $\Gamma$ equal to $\pi_1(S)$ or $F_n$ as in the introduction, we take $G \coloneqq [\Gamma, \Gamma]$ or $G \coloneqq K$ respectively. In either case, let $G_i$ be the lower central series of $G$, defined inductively by $G_1 = G$ and $G_{i+1} = [G, G_i]$. Consider the exact sequence

$$1 \to G_2 \to G \to G^{\mathrm{ab}} \to 1. \tag{1}$$

Centralizing (1) gives

$$1 \to G_2/G_3 \to G/G_3 \to G^{\mathrm{ab}} \to 1. \tag{2}$$

Since $G$ is free, taking (1) as a presentation for $G^{\mathrm{ab}}$, Hopf's formula gives that

$$G_2/G_3 \approx \textstyle\bigwedge^2 G^{\mathrm{ab}}.$$

$\mathrm{Aut}(\Gamma)$ acts on $\Gamma$, and thus on $G$, and the isomorphism $\nu\colon G_2/G_3 \approx \bigwedge^2 G^{\mathrm{ab}}$ respects the action of $\mathrm{Aut}(\Gamma)$ on both sides. In particular, conjugation by $\Gamma$ descends to an action on $G^{\mathrm{ab}}$ by $H = \Gamma/[\Gamma,\Gamma]$ or by $\mathbb{Z} = \Gamma/K$ respectively. In the case $G = [\Gamma,\Gamma]$, the fact that $\mathrm{Mag}_g$ acts trivially on $\Gamma/\Gamma^3$ implies that $\mathrm{Mag}_g$ acts trivially on $G^{\mathrm{ab}} = \Gamma^2/\Gamma^3$ and on $\bigwedge^2 G^{\mathrm{ab}}$. Similarly, in the case $G = K$, we have that $\mathrm{Bur}_n$ acts trivially on $G^{\mathrm{ab}}$ and on $\bigwedge^2 G^{\mathrm{ab}}$.

Let $f \in \mathrm{Mag}_g$ (resp. $f \in \mathrm{Bur}_n$) be given. For $x \in G^{\mathrm{ab}}$, pick any lift $\tilde{x} \in G$. Since $f$ acts trivially on both the quotient and kernel of (2), we see that $f(\tilde{x})\tilde{x}^{-1}$ lies in the kernel $G_2/G_3$, which we identify with $\bigwedge^2 G^{\mathrm{ab}}$ via the isomorphism above. One checks, exactly as in [J1], that

$$\delta_f\colon G^{\mathrm{ab}} \to \textstyle\bigwedge^2 G^{\mathrm{ab}}$$

defined by $\delta_f(x) \coloneqq f(\tilde{x})\tilde{x}^{-1}$ is a well-defined homomorphism; in fact, the resulting map $\delta_f$ is $\mathbb{Z}H$–linear (resp. $\mathbb{Z}[t,t^{-1}]$–linear) with respect to the conjugation action on $G^{\mathrm{ab}}$. This is equivalent to the claim that

$$\delta_f(\gamma x \gamma^{-1}) \equiv \gamma \delta_f(x) \gamma^{-1} \bmod G_3,$$

which can be checked as follows. The difference between the left and right side is

$$\big(f(\gamma x \gamma^{-1})\gamma x^{-1}\gamma^{-1}\big)\big(\gamma f(x) x^{-1}\gamma^{-1}\big)^{-1} = f(\gamma)f(x)f(\gamma)^{-1}\gamma f(x)^{-1}\gamma^{-1},$$

which is conjugate to $[\gamma^{-1}f(\gamma), f(x)]$. The condition on $f$ implies that $f(\gamma) \equiv \gamma \bmod G_2$, so $\gamma^{-1}f(\gamma) \in G_2$ and $[\gamma^{-1}f(\gamma), f(x)] \in G_3$ as desired.

One also checks, exactly as in [J1], that in the case $G = [\Gamma, \Gamma]$, defining the map $\Psi$ by $\Psi(f) := \delta_f$ gives a well-defined homomorphism;

$$\Psi\colon\ \mathrm{Mag}_g \to \mathrm{Hom}\left(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\right). \tag{3}$$

and, in the case $G = K$, defining $\Phi(g) := \delta_f$ gives a well-defined homomorphism:

$$\Phi\colon\ \mathrm{Bur}_n \to \mathrm{Hom}\left(G^{\mathrm{ab}}, \textstyle\bigwedge^2 G^{\mathrm{ab}}\right). \tag{4}$$

The homomorphisms $\Psi$ and $\Phi$ are equivariant with respect to the natural $\mathrm{Aut}(\Gamma)$-actions on the source and target.

## 3 Computing the image of $\Psi$

Let $S_{0,4}$ denote the 2-sphere with 4 open disks removed. A *lantern* in $S$ is an embedding $S_{0,4} \hookrightarrow S$. Consider the two simple closed curves $\alpha$ and $\beta$ and the three arcs $A_1, A_2$ and $A_3$ on $S_{0,4}$ given in Figure 1.

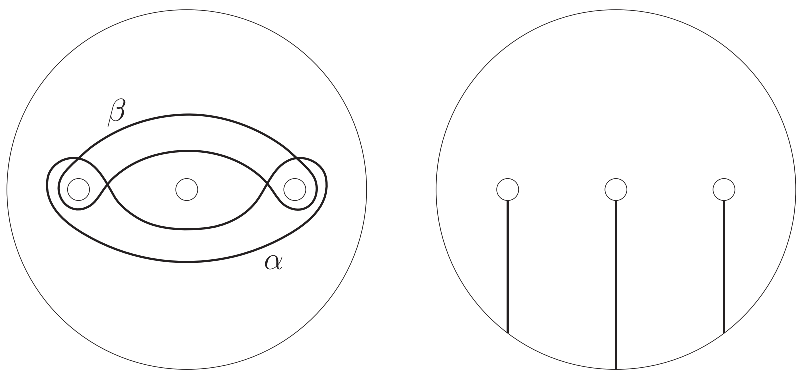


Figure 1: The simple closed curves $\alpha$ and $\beta$, and the arcs $A_1, A_2, A_3$.

One directly computes the action of $f := T_\alpha T_\beta^{-1}$ on $A_1$, $A_2$ and $A_3$, as follows (see Figure 2). Let $x$, $y$, and $z$ be the loops which begin with $A_1$, $A_2$ and $A_3$, respectively, go clockwise around the appropriate boundary component of $S_{0,4}$, then come back along the same arc $A_i$. Let $X, Y, Z$ be the inverses of $x, y, z$ in $\pi_1(S_{0,4})$. Then:

$$f(A_1) = xyXzxYXZA_1 = [xyX, z]A_1$$

$$f(A_2) = ZXzxA_2 = [Z, X]A_2$$

$$f(A_3) = ZXzxYXZxzxyXA_3 = [ZXz, xYX]A_3$$

Let $L$ be an embedding of a lantern in $S$ with the property that each of the four boundary curves of $L$ are separating in $S$.[1] In this case we can observe that $T_\alpha T_\beta^{-1} \in \mathrm{Mag}_g$, as follows.

[1]To formally identify $x, y, z$ with elements of $\Gamma = \pi_1(S)$, we choose a basepoint on $\partial S$, and arcs from this basepoint to $L$ meeting $L$ in one point. Since $f$ is the identity off of $L$, any ambiguity in the choice of these paths to $L$ does not affect the computation.

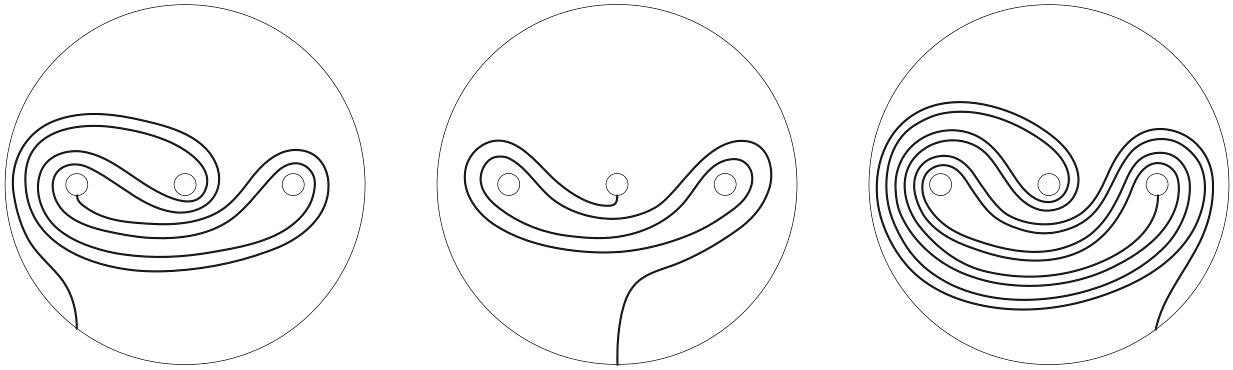

Figure 2: The arcs $f(A_1), f(A_2)$ and $f(A_3)$.

Note that the elements corresponding to $x, y, z$ all lie in $\Gamma^2$. Furthermore, $\Gamma = \pi_1(S)$ has a basis where each element $c$ is either disjoint from $L$, or else of the form $c = A\gamma A^{-1}$, where $A$ is an arc intersecting $L$ in some $A_i$ and $\gamma$ is a loop disjoint from $L$. In the former case the element $f = T_\alpha T_\beta^{-1}$ fixes $c$. In the latter case, assume for example that $A$ intersects $L$ in $A_2$; then we have

$$f(c) = f(A\gamma A^{-1}) = f(A)\gamma f(A)^{-1} = [Z, X]A\gamma A^{-1}[X, Z] = [Z, X]c[X, Z]$$

Since $x, y, z \in \Gamma^2$, we have $[Z, X] \in \Gamma^3$; thus $f(c) \equiv c \bmod \Gamma^3$. The same is true for $A_1$ and $A_3$, so we conclude that $f(c) \equiv c \bmod \Gamma^3$ for all elements of a basis for $\Gamma$, implying $T_\alpha T_\beta^{-1} \in \mathrm{Mag}_g$. Suzuki gave a more illuminating proof that elements of this form lie in $\mathrm{Mag}_g$ in [S2].

We are now ready to compute $\Psi$. For $a, b \in \Gamma$, we denote by $\{a, b\}$ the image of $[a, b] \in G$ in $G^{\mathrm{ab}}$ under the abelianization map.

**Proposition 3.1.** *Let $L$ be a lantern embedded in $S$ so that each of the four boundary curves of $L$ are separating in $S$. Let $a$ and $b$ be loops intersecting $L$ in $A_1$ and $A_2$. Then*

$$\Psi(T_\alpha T_\beta^{-1})(\{a, b\}) = (a-1)(b-1)\big[x \wedge z + y \wedge z\big] \tag{5}$$

Note that the right hand side of (5) is an element of $\bigwedge^2 G^{\mathrm{ab}}$, considered as a $\mathbb{Z}H$-module, and $a, b$ are taken to be elements of $H$.

*Proof.* As in the computation above, we have

$$f([a, b]) = [f(a), f(b)] = [wa, vb]$$

where

$$w = [[xyX, z], a] \quad \text{and} \quad v = [[Z, X], b].$$

From the assumption on the embedding of $L$ we have $x, y, z \in G$, and thus $w, v \in G_2$. We will use the following commutator identities, which hold in any group; we write ${}^x y$ for $xyx^{-1}$.

$$[wa, b] = {}^w[a, b]\ [w, b] \qquad\qquad [a, vb] = [a, v]\ {}^v[a, b]$$

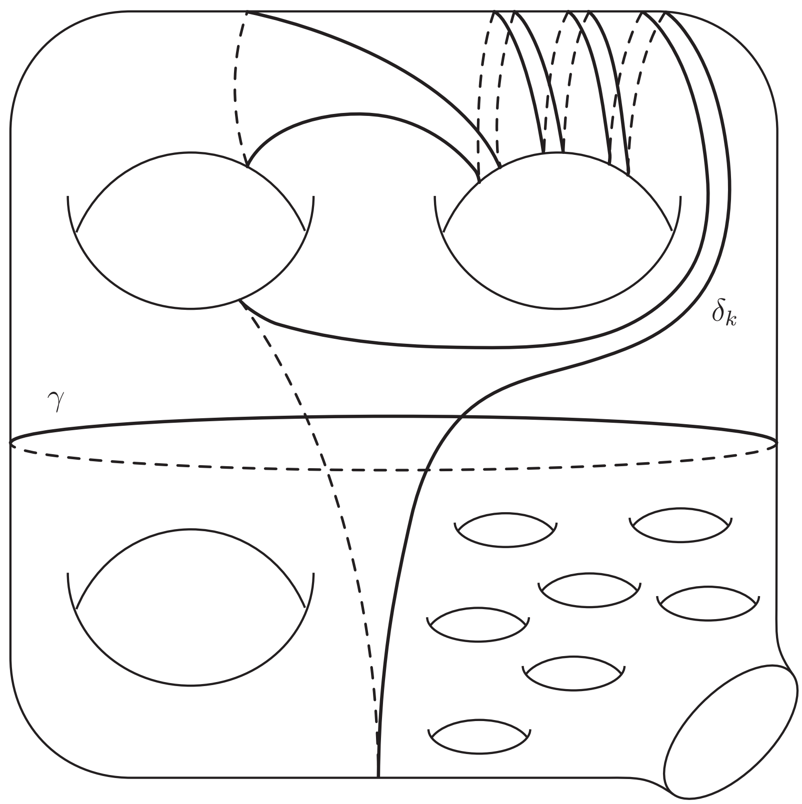


Figure 3: The curves $\gamma$ and $\delta_k$ for $k = 3$.

We then find that

$$[wa, vb] = {}^{w}[a, v]\ {}^{wv}[a, b]\ [w, v]\ {}^{v}[w, b]$$

Note that the second term lies in $G$, the first and fourth in $G_2$, and the third in $G_3$.

We want to compute $f([a, b])[a, b]^{-1}$ as an element of $G_2/G_3$. Note that $[w, v] \equiv 0 \bmod G_3$, and that conjugating an element of $G$ by an element of $G_2$ is a trivial operation modulo $G_3$. Finally, since $[[a, b], [w, b]] \in G_3$, we can move $[a, b]$ to the right to cancel $[a, b]^{-1}$. We thus obtain

$$\begin{aligned} f([a, b])[a, b]^{-1} &= {}^{w}[a, v]\ {}^{wv}[a, b]\ [w, v]\ {}^{v}[w, b]\ [a, b]^{-1} \\ &\equiv [a, v][a, b][w, b][a, b]^{-1} \bmod G_3 \\ &\equiv [a, v][w, b] \bmod G_3. \end{aligned}$$

Recall that action of $\Gamma$ on $\Gamma$ by conjugation decends to a $\mathbb{Z}H$ action on $G^{\mathrm{ab}}$. Recall from above the isomorphism $\nu\colon G_2/G_3 \to \bigwedge^2 G^{\mathrm{ab}}$. Since the homology class of $x$ is trivial in $H$, we have

$$\nu([xyX, z]) = y \wedge z \quad \text{and} \quad \nu([Z, X]) = z \wedge x.$$

It follows that

$$\nu(w) = \nu([[xyX, z], a]) = (1 - a)y \wedge z$$

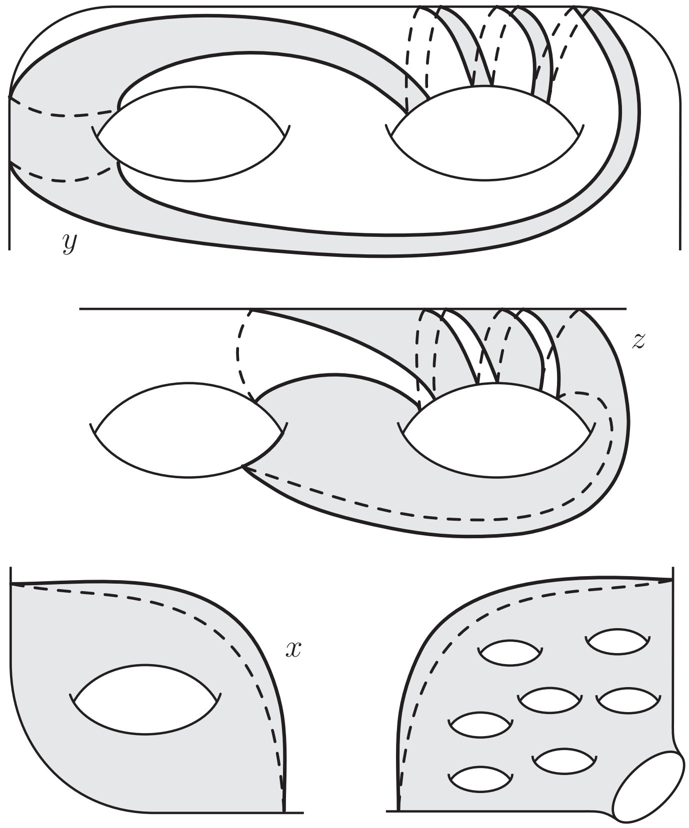


Figure 4: The boundary curves of $L_k$; the subsurfaces cut off by these curves are shaded.

and

$$\nu(v) = \nu([[Z, X], b]) = (1 - b)z \wedge x.$$

We therefore have that

$$\nu([a, v][w, b]) = (a - 1)v - (b - 1)w = (a - 1)(1 - b)z \wedge x - (b - 1)(1 - a)y \wedge z.$$

We conclude that

$$\Psi(T_\alpha T_\beta^{-1})(\{a, b\}) = (a - 1)(b - 1)\big[x \wedge z + y \wedge z\big]$$

as desired. $\square$

**Theorem 3.2.** *The image of $\Psi$ has infinite rank for $g \geq 3$.*

*Proof.* Let $\gamma$ and $\delta_k$ be the curves depicted in Figure 3. The figure depicts the case $k = 3$; in general $\delta_k$ has $k$ twists around the upper right handle. (Specifically, the curve $\delta_k$ is equal to $T_{a_3}^k(\delta_0)$, where $a_3$ is as in Figure 5.) The regular neighborhood of $\gamma \cup \delta_k$ is a lantern $L_k$, and we fix an identification of $L_k$ with our reference lantern $L$ by specifying that $\gamma$ and $\delta_k$ should

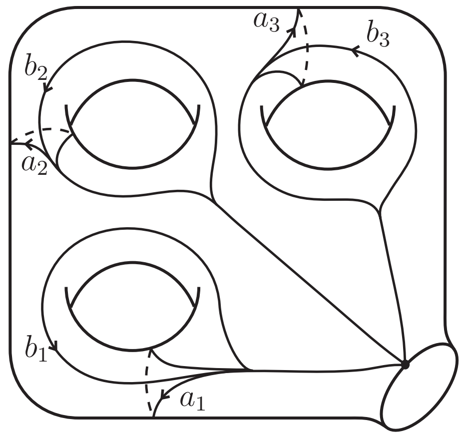


Figure 5: A basis for $\pi_1(S_{g,1})$.

correspond to $xy$ and $yz$ respectively. Let $f_k \in \mathrm{Mag}_g$ be the element corresponding under this identification to the mapping class $T_\alpha T_\beta^{-1}$ on $L$; it is easy to check using the lantern relation that $f_k$ is in fact $[T_\gamma^{-1}, T_{\delta_k}^{-1}]$. We will show that the images $\Psi(f_k)$ are linearly independent (over $\mathbb{Z}$).

The boundary curves of $L_k$ are depicted in Figure 4. With the basis $a_1, b_1, \ldots, a_g, b_g$ for $\pi_1(S_{g,1})$ as illustrated in Figure 5, we see that as curves $x$, $y$ and $z$ can be represented by $[a_1, b_1]$, $[a_2, b_3a_3^kb_2]$, and $[b_2a_2^{-1}b_2^{-1}a_3, b_3a_3^k]$ respectively. As based loops, we actually have the conjugate $z = {}^c[b_2a_2^{-1}b_2^{-1}a_3, b_3a_3^k]$, where $c = [b_3, a_3][b_2, a_2]a_2$. Note that with this representative for $z$, we have $xyz = [a_1, b_1][a_2, b_2][a_3, b_3]$, the fourth boundary curve in Figure 4.

Note that $a_1$ and $a_2$ intersect each $L_k$ in arcs corresponding to $A_1$ and $A_2$. Thus by Proposition 3.1, we have that

$$\Psi(f_k)(\{a_1, a_2\}) = (a_1 - 1)(a_2 - 1)\big[\big(\{a_1, b_1\} + \{a_2, b_3a_3^kb_2\}\big) \wedge a_2\{b_2a_2^{-1}b_2^{-1}a_3, b_3a_3^k\}\big]$$

Denote this element of $\bigwedge^2 G^{\mathrm{ab}}$ by $\alpha_k$. We now check that $\{\alpha_k\}$ is linearly independent as follows. There is a standard embedding $G^{\mathrm{ab}} \hookrightarrow (\mathbb{Z}H)^{2g}$ given by sending the class $[x]$ to $(\partial x/\partial z_1, \ldots, \partial x/\partial z_n)$, where $\{z_i\}$ is our basis for $F_n$ and where $\partial/\partial z_i$ are the Fox derivatives (see e.g. [CP] for a detailed explanation of this embedding). The only property of this embedding that we will need is that the components that make up $\alpha_k$ are mapped as follows by the embedding. Here the $A_i$ and $B_i$ make up a basis for $(\mathbb{Z}H)^{2g}$.

$$\begin{aligned}
\{a_1, b_1\} &\mapsto \quad (1 - b_1)A_1 - (1 - a_1)B_1 \\
\{a_2, b_3a_3^kb_2\} &\mapsto \quad (1 - b_3a_3^kb_2)A_2 \\
&\qquad -(1 - a_2)\big(B_3 + b_3(1 + \cdots + a_3^{k-1})A_3 + b_3a_3^kB_2\big) \\
\{b_2a_2^{-1}b_2^{-1}a_3, b_3a_3^k\} &\mapsto \quad (1 - b_3a_3^k)\big((1 - a_2^{-1})B_2 - a_2^{-1}b_2A_2 + a_2^{-1}A_3\big) \\
&\qquad -(1 - a_2^{-1}a_3)\big(B_3 + b_3(1 + \cdots + a_3^{k-1})A_3\big)
\end{aligned}$$

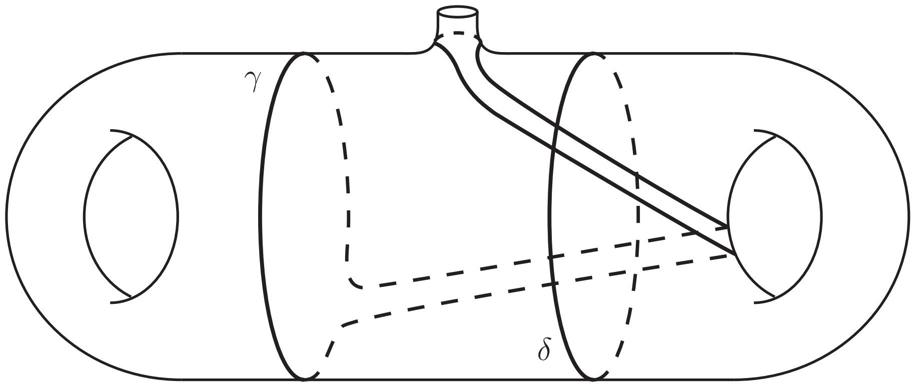


Figure 6: The commutator $[T_\gamma, T_\delta]$ lies in $\mathrm{Mag}_2$.

By expanding out $\alpha_k$, we see that $\alpha_N$ is the only such element which contains the term $A_1 \wedge b_3 a_3^N B_2$ with nonzero coefficient; it follows that the $\alpha_k$ are linearly independent, as desired. □

As the image of $\Psi$ is abelian, Theorem 3.2 immediately implies Theorem 1.1 for $g \geq 3$. Note that the proof of Theorem 3.2 used in an essential way that $g \geq 3$. So in order to complete the proof of Theorem 1.1, we need another argument when $g = 2$.

**Theorem 3.3.** *$H_1(\mathrm{Mag}_2)$ has infinite rank.*

*Proof.* Suzuki showed that the element $f = [T_\gamma, T_\delta]$ is in $\mathrm{Mag}_2$ for $\gamma$ and $\delta$ as in Figure 6; in particular $\mathrm{Mag}_2$ is nontrivial. Let $S_2$ be a closed surface of genus 2; we denote by $\mathcal{I}_{2,*}$ the Torelli group of $S_2$ with respect to a marked point $*$, and by $\mathcal{I}_2$ the Torelli group of the closed surface $S_2$. By Johnson [J2], we have the exact sequence

$$1 \to \mathbb{Z} \to \mathcal{I}_{2,1} \xrightarrow{p} \mathcal{I}_{2,*} \to 1,$$

where the kernel is generated by a twist $T_\omega$ around the boundary $\omega = \partial S_2$. It is easy to check that the action of $T_\omega$ on $\pi_1(S_{2,1})$ is conjugation by $\omega$; since $\omega \notin \Gamma^3$, we see that $T_\omega \notin \mathrm{Mag}_2$. It follows that $p$ restricts to an isomorphism between $\mathrm{Mag}_2$ and a subgroup $p(\mathrm{Mag}_2) < \mathcal{I}_{2,*}$.

Again by Johnson [J2], we have the exact sequence

$$1 \to \Lambda \to \mathcal{I}_{2,*} \xrightarrow{\pi} \mathcal{I}_2 \to 1,$$

where $\Lambda \approx \pi_1(S_2, *)$; note that $\mathcal{I}_{2,*}$ acts on $\pi_1(S_2, *)$, and the restriction to $\Lambda$ is just the action by conjugation. Mess [Me] proved that $\mathcal{I}_2$ is free of infinite rank. It is easy to see from Figure 6 that $f \in \ker \pi = \Lambda$. We use the following well-known lemma.

**Lemma 3.4.** *Any nontrivial infinite index normal subgroup of a surface group or free group is an infinite rank free group.*

If $\pi \circ p(\mathrm{Mag}_2) < \mathcal{I}_2 \approx F_\infty$ is nontrivial, then by Lemma 3.4, $\mathrm{Mag}_2$ surjects to the infinite rank free group $\pi \circ p(\mathrm{Mag}_2)$, and we are done.

Suppose that $p(\mathrm{Mag}_2) \subset \ker \pi = \Lambda$. Any $\varphi \in \mathrm{Mag}_2$ acts trivially on $\Gamma/\Gamma^3$; thus $p(\varphi)$ acts trivially on $\pi_1(S_2)/\pi_1(S_2)^3$. Since the action of $\Lambda$ is by conjugation, this implies that $p(\varphi)$ lies in $\Lambda^3$. Thus $p(\mathrm{Mag}_2)$ has infinite index in $\Lambda$, and so by Lemma 3.4, $p(\mathrm{Mag}_2) \approx \mathrm{Mag}_2$ is an infinite rank free group. $\square$

Theorem 1.1, and hence Corollary 1.2, follows immediately from Theorems 3.2 and 3.3.

**Remark.** One can check by explicit computation that for Suzuki's element $f \in \mathrm{Mag}_2$ above, $\Psi(f) = 0$. It would be interesting to know whether $\Psi$ in fact vanishes on $\mathrm{Mag}_2$.

## 4 Computing the image of $\Phi$

The kernel $K$ of the map from $F_n = \langle x_1, \ldots, x_n \rangle$ to $\mathbb{Z} = \langle t \rangle$ which sends each $x_i \mapsto t$ is normally generated by the elements $x_i x_j^{-1}$. If we set $x_{i,k} := x_1^k x_i x_1^{-k-1}$ for $i \neq 1$ and $k \in \mathbb{Z}$, then $\{x_{i,k}\}$ gives a basis for $K$ as a free group. As above, the conjugation of $K$ by $F_n$ descends to a $\mathbb{Z}[t, t^{-1}]$ action on $K^{\mathrm{ab}}$. With respect to this action we have $x_{i,k} = t^k x_{i,0}$, and thus $K^{\mathrm{ab}}$ is a free $\mathbb{Z}[t, t^{-1}]$–module with basis $\{y_i = x_{i,0}\}_{i \neq 1}$.

The braid group $B_n$ has generators $\sigma_1, \ldots, \sigma_{n-1}$; the action of $\sigma_i$ on $F_n$ sends $x_i \mapsto x_i x_{i+1} x_i^{-1}$, $x_{i+1} \mapsto x_i$, and fixes the other generators. The action of $B_n$ on $K^{\mathrm{ab}}$ commutes with the $\mathbb{Z}[t, t^{-1}]$ action.

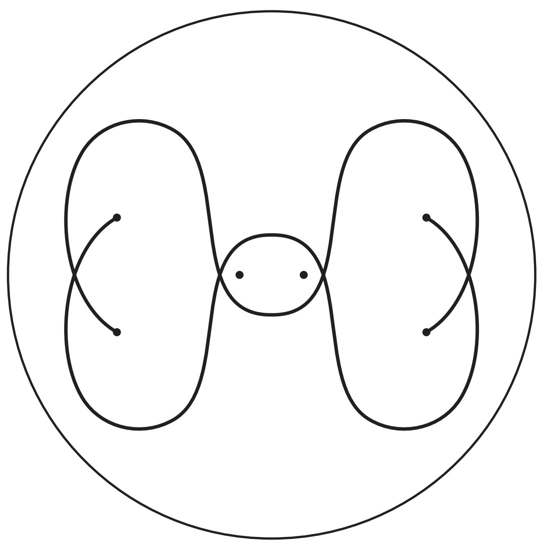

Figure 7: The two arcs defining Bigelow's element $\phi_B$.

**Theorem 4.1.** *The image of $\Phi$ has infinite rank for $n \geq 6$.*

*Proof.* The element of the kernel found by Bigelow in [Big] is the commutator of the half-twists along the arcs displayed in Figure 7. In terms of the Artin generators, this is

$$\phi_B = [\psi_1 \sigma_3^{-1} \psi_1^{-1}, \psi_2 \sigma_3^{-1} \psi_2], \qquad \text{where } \psi_1 = \sigma_4 \sigma_5^{-1} \sigma_2^{-1} \sigma_1 \quad \text{and } \psi_2 = \sigma_4^{-1} \sigma_5^2 \sigma_2 \sigma_1^{-1}.$$

In Appendix A, we give the computation of $\alpha := \Phi(\phi_B)([x_2x_1^{-1}]) = \Phi(\phi_B)(y_2)$; it has 262 terms. The only fact about $\alpha$ that we will need is that its highest term of the form $y_2 \wedge t^k y_4$ is $-2y_2 \wedge t^3 y_4$, and its highest term of the form $y_2 \wedge t^k y_5$ is $+2y_2 \wedge t^2 y_5$ (these terms are set in boxes in the appendix).

It is easy to check that

$$\sigma_4^2(x_4) = x_4x_5x_4x_5^{-1}x_4^{-1}$$

$$\sigma_4^2(x_5) = x_5x_4x_5^{-1}$$

$$\sigma_4^2(x_i) = x_i \quad \text{for } i \neq 4, 5.$$

By induction, for $k \geq 1$ we have

$$\sigma_4^{2k}(x_4) = (x_4x_5)^k x_4 (x_4x_5)^{-k}$$

$$\sigma_4^{2k}(x_5) = (x_4x_5)^{k-1} x_4x_5x_4^{-1}(x_4x_5)^{k-1}$$

$$\sigma_4^{2k}(x_i) = x_i \quad \text{for } i \neq 4, 5.$$

The action of $\sigma_4^{2k}$ on $K^{\text{ab}}$ in terms of our basis is thus given by:

$$\begin{array}{rcl} y_4 & \mapsto & (1 - t + t^2 - \cdots - t^{k-1} + t^k)y_4 \quad + \quad (t - t^2 + \cdots + t^{k-1} - t^k)y_5 \\ y_5 & \mapsto & (1 - t + t^2 - \cdots - t^{k-1})y_4 \qquad\quad + \quad (t - t^2 + \cdots + t^{k-1})y_5 \\ y_i & \mapsto & y_i \quad \text{for } i \neq 4, 5 \end{array}$$

Now for $k \geq 0$ set

$$\alpha_k := \Phi(\sigma_4^{2k}\phi_B\sigma_4^{-2k})(y_2).$$

By the equivariance of $\Phi$, and since $\sigma_4$ fixes $y_2$, we have $\alpha_k = \sigma_4^{2k} \cdot \alpha$. From the action of $\sigma_4^{2k}$ on $K^{\text{ab}}$, we can see that the highest term in $\alpha_N$ of the form $y_2 \wedge t^k y_4$ will be $-2y_2 \wedge t^{3+N} y_4$. Thus $\alpha_N$ is not contained in the span of $\{\alpha_1, \ldots, \alpha_{N-1}\}$; it follows that the $\alpha_k$ are linearly independent over $\mathbb{Z}$, and thus the image of $\Phi$ has infinite rank. $\square$

Theorem 1.3 follows immediately.

# A Appendix

The following computation was made, with the method explained in Section 4, with the help of *Mathematica*. A *Mathematica* notebook implementing these computations can be found at:

`http://math.uchicago.edu/~tchurch/infinitegeneration.html`

The output of this notebook is $\Phi(\phi_B)(y_2)$, which is:

$$\begin{array}{rrrrr} -t^{-3}y_2 \wedge t^{-2}y_2 & +t^{-3}y_2 \wedge t^{-1}y_2 & -t^{-3}y_2 \wedge y_2 & -t^{-2}y_2 \wedge y_2 & +t^{-1}y_2 \wedge y_2 \\ +t^{-2}y_2 \wedge ty_2 & +t^{-1}y_2 \wedge ty_2 & -2y_2 \wedge t^2y_2 & +ty_2 \wedge t^3y_2 & +t^2y_2 \wedge t^3y_2 \\ -t^3y_2 \wedge t^4y_2 & +t^{-3}y_2 \wedge t^{-4}y_3 & -t^{-2}y_2 \wedge t^{-4}y_3 & -t^{-3}y_2 \wedge t^{-3}y_3 & +t^{-1}y_2 \wedge t^{-3}y_3 \end{array}$$

$$
\begin{array}{rrrrr}
+t^{-2}y_2\wedge t^{-2}y_3 & -t^{-1}y_2\wedge t^{-2}y_3 & +t^{-3}y_2\wedge t^{-1}y_3 & -y_2\wedge t^{-1}y_3 & +ty_2\wedge t^{-1}y_3 \\
-t^{2}y_2\wedge t^{-1}y_3 & -2t^{-2}y_2\wedge y_3 & +t^{3}y_2\wedge y_3 & +t^{-1}y_3\wedge y_3 & +2t^{-1}y_2\wedge ty_3 \\
-t^{-1}y_3\wedge ty_3 & -2y_2\wedge t^{2}y_3 & -t^{4}y_2\wedge t^{2}y_3 & +t^{-1}y_3\wedge t^{2}y_3 & +ty_2\wedge t^{3}y_3 \\
+t^{4}y_2\wedge t^{3}y_3 & -y_3\wedge t^{3}y_3 & +ty_3\wedge t^{3}y_3 & -t^{2}y_3\wedge t^{3}y_3 & +t^{-3}y_2\wedge t^{-3}y_4 \\
-t^{-2}y_2\wedge t^{-3}y_4 & -t^{-3}y_2\wedge t^{-2}y_4 & +t^{-1}y_2\wedge t^{-2}y_4 & +t^{-2}y_2\wedge t^{-1}y_4 & -t^{-1}y_2\wedge t^{-1}y_4 \\
+t^{-3}y_2\wedge y_4 & -y_2\wedge y_4 & +ty_2\wedge y_4 & -t^{2}y_2\wedge y_4 & -y_3\wedge y_4 \\
+ty_3\wedge y_4 & -t^{2}y_3\wedge y_4 & -2t^{-2}y_2\wedge ty_4 & +t^{3}y_2\wedge ty_4 & +t^{-1}y_3\wedge ty_4 \\
+t^{3}y_3\wedge ty_4 & +y_4\wedge ty_4 & +2t^{-1}y_2\wedge t^{2}y_4 & -t^{-1}y_3\wedge t^{2}y_4 & -t^{3}y_3\wedge t^{2}y_4 \\
-y_4\wedge t^{2}y_4 & \boxed{-2y_2\wedge t^{3}y_4} & -t^{4}y_2\wedge t^{3}y_4 & +t^{-1}y_3\wedge t^{3}y_4 & +t^{3}y_3\wedge t^{3}y_4 \\
+y_4\wedge t^{3}y_4 & +ty_2\wedge t^{4}y_4 & +t^{4}y_2\wedge t^{4}y_4 & -y_3\wedge t^{4}y_4 & +ty_3\wedge t^{4}y_4 \\
-t^{2}y_3\wedge t^{4}y_4 & -ty_4\wedge t^{4}y_4 & +t^{2}y_4\wedge t^{4}y_4 & -t^{3}y_4\wedge t^{4}y_4 & +t^{-3}y_2\wedge t^{-3}y_5 \\
-t^{-2}y_2\wedge t^{-3}y_5 & +t^{-3}y_2\wedge t^{-2}y_5 & -t^{-2}y_2\wedge t^{-2}y_5 & +y_2\wedge t^{-2}y_5 & -t^{-4}y_3\wedge t^{-2}y_5 \\
+t^{-3}y_3\wedge t^{-2}y_5 & -t^{-1}y_3\wedge t^{-2}y_5 & -t^{-3}y_4\wedge t^{-2}y_5 & +t^{-2}y_4\wedge t^{-2}y_5 & -y_4\wedge t^{-2}y_5 \\
-t^{-3}y_5\wedge t^{-2}y_5 & -2t^{-3}y_2\wedge t^{-1}y_5 & +t^{-1}y_2\wedge t^{-1}y_5 & +y_2\wedge t^{-1}y_5 & -ty_2\wedge t^{-1}y_5 \\
+t^{-4}y_3\wedge t^{-1}y_5 & -t^{-2}y_3\wedge t^{-1}y_5 & +2y_3\wedge t^{-1}y_5 & +t^{-3}y_4\wedge t^{-1}y_5 & -t^{-1}y_4\wedge t^{-1}y_5 \\
+2ty_4\wedge t^{-1}y_5 & +t^{-3}y_5\wedge t^{-1}y_5 & +t^{-3}y_2\wedge y_5 & +2t^{-2}y_2\wedge y_5 & -2t^{-1}y_2\wedge y_5 \\
-y_2\wedge y_5 & -t^{2}y_2\wedge y_5 & -t^{-3}y_3\wedge y_5 & +t^{-2}y_3\wedge y_5 & -y_3\wedge y_5 \\
-ty_3\wedge y_5 & -t^{2}y_3\wedge y_5 & -t^{-2}y_4\wedge y_5 & +t^{-1}y_4\wedge y_5 & -ty_4\wedge y_5 \\
-t^{2}y_4\wedge y_5 & -t^{3}y_4\wedge y_5 & +t^{-1}y_5\wedge y_5 & -t^{-3}y_2\wedge ty_5 & -t^{-1}y_2\wedge ty_5 \\
+y_2\wedge ty_5 & +ty_2\wedge ty_5 & +t^{3}y_2\wedge ty_5 & +t^{-1}y_3\wedge ty_5 & -y_3\wedge ty_5 \\
+2ty_3\wedge ty_5 & +t^{3}y_3\wedge ty_5 & +y_4\wedge ty_5 & -ty_4\wedge ty_5 & +2t^{2}y_4\wedge ty_5 \\
+t^{4}y_4\wedge ty_5 & -t^{-2}y_5\wedge ty_5 & -y_5\wedge ty_5 & +t^{-2}y_2\wedge t^{2}y_5 & -t^{-1}y_2\wedge t^{2}y_5 \\
\boxed{+2y_2\wedge t^{2}y_5} & -t^{2}y_2\wedge t^{2}y_5 & +t^{3}y_2\wedge t^{2}y_5 & -t^{-1}y_3\wedge t^{2}y_5 & -t^{2}y_3\wedge t^{2}y_5 \\
-y_4\wedge t^{2}y_5 & -t^{3}y_4\wedge t^{2}y_5 & +t^{-1}y_5\wedge t^{2}y_5 & -2y_5\wedge t^{2}y_5 & +ty_5\wedge t^{2}y_5 \\
-ty_2\wedge t^{3}y_5 & -t^{2}y_2\wedge t^{3}y_5 & -t^{4}y_2\wedge t^{3}y_5 & +y_3\wedge t^{3}y_5 & +ty_4\wedge t^{3}y_5 \\
+ty_5\wedge t^{3}y_5 & +t^{2}y_5\wedge t^{3}y_5 & +t^{2}y_2\wedge t^{4}y_5 & +t^{3}y_2\wedge t^{4}y_5 & -ty_3\wedge t^{4}y_5 \\
+t^{3}y_3\wedge t^{4}y_5 & -t^{2}y_4\wedge t^{4}y_5 & +t^{4}y_4\wedge t^{4}y_5 & -t^{2}y_5\wedge t^{4}y_5 & -t^{3}y_5\wedge t^{4}y_5 \\
-t^{3}y_2\wedge t^{5}y_5 & +t^{2}y_3\wedge t^{5}y_5 & -t^{3}y_3\wedge t^{5}y_5 & +t^{3}y_4\wedge t^{5}y_5 & -t^{4}y_4\wedge t^{5}y_5 \\
+t^{3}y_5\wedge t^{5}y_5 & -t^{-3}y_2\wedge t^{-3}y_6 & +t^{-2}y_2\wedge t^{-3}y_6 & -t^{-2}y_5\wedge t^{-3}y_6 & +t^{-1}y_5\wedge t^{-3}y_6 \\
+t^{-3}y_2\wedge t^{-2}y_6 & -t^{-1}y_2\wedge t^{-2}y_6 & +t^{-2}y_5\wedge t^{-2}y_6 & -y_5\wedge t^{-2}y_6 & +t^{-3}y_2\wedge t^{-1}y_6 \\
-t^{-2}y_2\wedge t^{-1}y_6 & +y_2\wedge t^{-1}y_6 & -t^{-4}y_3\wedge t^{-1}y_6 & +t^{-3}y_3\wedge t^{-1}y_6 & -t^{-1}y_3\wedge t^{-1}y_6 \\
-t^{-3}y_4\wedge t^{-1}y_6 & +t^{-2}y_4\wedge t^{-1}y_6 & -y_4\wedge t^{-1}y_6 & -t^{-3}y_5\wedge t^{-1}y_6 & +ty_5\wedge t^{-1}y_6 \\
+t^{-3}y_6\wedge t^{-1}y_6 & -t^{-2}y_6\wedge t^{-1}y_6 & -t^{-3}y_2\wedge y_6 & -t^{-2}y_2\wedge y_6 & +t^{-1}y_2\wedge y_6 \\
+2y_2\wedge y_6 & -2ty_2\wedge y_6 & +t^{2}y_2\wedge y_6 & +t^{-3}y_3\wedge y_6 & -t^{-2}y_3\wedge y_6 \\
-t^{-1}y_3\wedge y_6 & +3y_3\wedge y_6 & -ty_3\wedge y_6 & +t^{2}y_3\wedge y_6 & +t^{-2}y_4\wedge y_6 \\
-t^{-1}y_4\wedge y_6 & -y_4\wedge y_6 & +3ty_4\wedge y_6 & -t^{2}y_4\wedge y_6 & +t^{3}y_4\wedge y_6 \\
-y_5\wedge y_6 & +ty_5\wedge y_6 & -2t^{2}y_5\wedge y_6 & -t^{-2}y_6\wedge y_6 & +t^{-3}y_2\wedge ty_6 \\
+t^{-2}y_2\wedge ty_6 & -y_2\wedge ty_6 & -ty_2\wedge ty_6 & -t^{3}y_2\wedge ty_6 & -t^{-1}y_3\wedge ty_6 \\
+y_3\wedge ty_6 & -2ty_3\wedge ty_6 & -t^{3}y_3\wedge ty_6 & -y_4\wedge ty_6 & +ty_4\wedge ty_6 \\
-2t^{2}y_4\wedge ty_6 & -t^{4}y_4\wedge ty_6 & +t^{-2}y_5\wedge ty_6 & +t^{-1}y_5\wedge ty_6 & +t^{2}y_5\wedge ty_6 \\
+t^{3}y_5\wedge ty_6 & +t^{-1}y_6\wedge ty_6 & +2y_6\wedge ty_6 & -t^{-2}y_2\wedge t^{2}y_6 & -t^{-1}y_2\wedge t^{2}y_6 \\
+t^{2}y_2\wedge t^{2}y_6 & +t^{-1}y_3\wedge t^{2}y_6 & +t^{2}y_3\wedge t^{2}y_6 & +t^{3}y_3\wedge t^{2}y_6 & +y_4\wedge t^{2}y_6 \\
+t^{3}y_4\wedge t^{2}y_6 & +t^{4}y_4\wedge t^{2}y_6 & -t^{-1}y_5\wedge t^{2}y_6 & +ty_5\wedge t^{2}y_6 & -t^{2}y_5\wedge t^{2}y_6 \\
-t^{4}y_5\wedge t^{2}y_6 & -2y_6\wedge t^{2}y_6 & -ty_6\wedge t^{2}y_6 & +2y_2\wedge t^{3}y_6 & +t^{4}y_2\wedge t^{3}y_6
\end{array}
$$

$$\begin{array}{rrrrr} -t^{-1}y_3 \wedge t^3y_6 & -t^3y_3 \wedge t^3y_6 & -y_4 \wedge t^3y_6 & -t^4y_4 \wedge t^3y_6 & -y_5 \wedge t^3y_6 \\ -t^2y_5 \wedge t^3y_6 & +t^5y_5 \wedge t^3y_6 & +y_6 \wedge t^3y_6 & +t^2y_6 \wedge t^3y_6 & -ty_2 \wedge t^4y_6 \\ -t^2y_2 \wedge t^4y_6 & -t^4y_2 \wedge t^4y_6 & +y_3 \wedge t^4y_6 & +ty_4 \wedge t^4y_6 & +ty_5 \wedge t^4y_6 \\ +t^2y_5 \wedge t^4y_6 & +t^4y_5 \wedge t^4y_6 & -t^5y_5 \wedge t^4y_6 & -ty_6 \wedge t^4y_6 & +t^3y_2 \wedge t^5y_6 \\ -t^2y_3 \wedge t^5y_6 & +t^3y_3 \wedge t^5y_6 & -t^3y_4 \wedge t^5y_6 & +t^4y_4 \wedge t^5y_6 & -t^3y_5 \wedge t^5y_6 \\ +t^3y_6 \wedge t^5y_6 & -t^4y_6 \wedge t^5y_6 & & & \end{array}$$

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